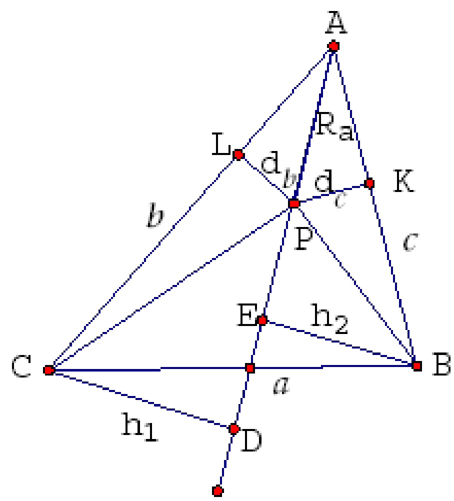
Basic inequalities with distances.
(**Extraction from the notes on the theme "Inequalities with distances", Arkady Alt**)
It is well known inequality usually used as Lemma in the proof of
Erdos-Mordell inequality, but important by itself, inequality

(**B1**) $\boxed{aR_a \geq bd_c + cd_b.}$



Pic.1

Proof.

Let PK, PM, PL are perpendiculars from P to sides BC, CA, AB respectively. Then $d_a = |PK|$,
$d_b = PL,\ d_c := PM.$

Let LE and MQ be perpendiculars to $\overleftrightarrow{KP}$. Since $\angle MPF = \angle KBM = \angle ABC$ and $\angle LPE = \angle LCK = \angle ACB$ then $MF = d_c \sin\angle ABC$ and $LF = d_b \sin\angle ACB$ and we obtain $MF + LE \leq MQ + LQ = ML \Leftrightarrow d_c \sin\angle ABC + d_b \sin\angle ACB \leq ML.$

Since $\angle AMP = \angle ALP = 90^\circ$ then $R_a = AP$ is diameter of circumcircle for quadrilateral $ALPM$
then by sin-theorem $R_a = \dfrac{ML}{\sin\angle CAB} \Leftrightarrow ML = R_a \sin\angle CAB.$

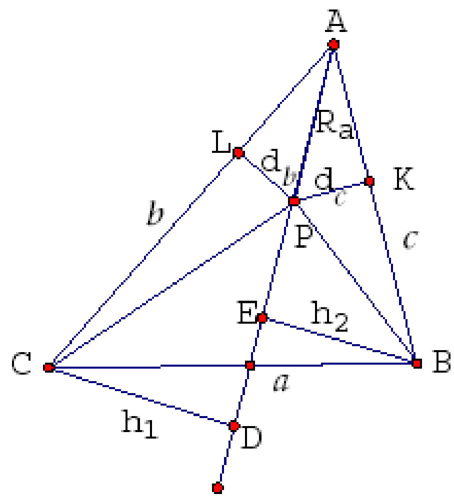
Thus $R_a \sin\angle CAB \geq d_c \sin\angle ABC + d_b \sin\angle ACB$ and multiplying both sides of this inequality by
$2R,$ where R is circumradius of triangle $ABC,$ we finally obtain $aR_a \geq bd_c + cd_b.$

Equality condition in inequality (**B1**) holds iff $EQ = 0$ and $FQ = 0,$ i.e. iff $ML \parallel BC.$

Another basic inequality.

Some times can be useful similar to (**B1**) inequality

(**B2**) $\boxed{aR_a \geq bd_b + cd_c}$.



Pic.2.

Proof.

From similarity $\triangle PLA \sim \triangle CDA$ and $\triangle PKA \sim \triangle BEA$ we obtain,respectively:

$\frac{d_b}{R_a} = \frac{h_1}{b} \Leftrightarrow h_1R_a = bd_b$ and $\frac{d_c}{R_a} = \frac{h_2}{c} \Leftrightarrow h_2R_a = cd_c$.

Hence, $R_a(h_1 + h_2) = bd_b + cd_c$ and since $a \geq h_1 + h_2$ we finally obtain

$aR_a \geq R_a(h_1 + h_2) = bd_b + cd_c$.

Equality holds iff $AP \perp BC$.